

Moduli of products of curves

Michael A. van Opstall

Abstract

Some technical results on the deformations of varieties of general type and on permanence of semi-log-canonical singularities are proved. These results are applied to show that the connected component of the moduli space of stable surfaces containing the moduli point of a product of stable curves is the product of the moduli spaces of the curves, assuming the curves have different genera. An application of this result shows that even after compactifying the moduli space and fixing numerical invariants, the moduli spaces are still very disconnected.

The main result of this article is the construction of several connected, irreducible components of the moduli space of stable surfaces. These components parameterize products of stable curves. They are constructed from the corresponding moduli spaces of stable curves. The essential results are that products of stable curves are stable surfaces, and all infinitesimal deformations of a product of stable varieties (of any dimension) come from the deformations of the factors. In particular, this gives a proof that even after fixing certain topological invariants, the resulting moduli space may have arbitrarily many components.

These examples also show that these components of the moduli space of smooth minimal surfaces of general type are not joined in the moduli space of stable surfaces.

All schemes are defined over the field \mathbf{C} . A *variety* will be a connected, reduced, and separated scheme of finite type, not necessarily assumed irreducible. A *family* will be a flat morphism of varieties. The deformation theory results used can be found in [Vis]; the base space of a miniversal deformation will be called the *Kuranishi space*, following the convenient terminology from analytic geometry.

I thank Sándor Kovács and E. Lee Stout for useful conversations. Thanks also to an anonymous reader who noted some simplifications from an earlier version.

1 Semi-log-canonical singularities and moduli of stable surfaces

First a preliminary remark: if X is a S_2 variety which is Gorenstein in codimension 1, then the extension of the dualizing sheaf of the Gorenstein locus (which is locally free) is a reflexive sheaf on X which corresponds to a Weil divisor K_X . X is called **Q**-Gorenstein if some multiple of K_X is Cartier. Using these definitions, one may define the class of singularities to be studied.

Definition 1.1. A variety X is said to have *semi-log-canonical* (slc) singularities if

1. X is **Q**-Gorenstein;
2. X is S_2 ;
3. X has at worst normal crossing singularities is codimension 1;
4. there exists a good desingularization $f : Y \rightarrow X$ such that in the formula

$$K_Y \equiv f^*K_X + \sum a_i E_i$$

all of the a_i are positive.

⁰University of Washington, opstall@math.washington.edu.

The moduli space of stable surfaces with fixed Hilbert polynomial is difficult to define. In fact, there are many different definitions which make sense but lead to nonisomorphic moduli spaces. The original definition (as well as the definition of slc singularities) appeared in [KSB88]. Later Kollár amended the conditions that a family of stable varieties should satisfy in [Kol90].

In the case of Gorenstein varieties, these subtleties do not occur, and the moduli functors do not differ. However, it is only in special cases that the moduli space of minimal surfaces of general type can be compactified by adding stable surfaces with only Gorenstein singularities to the moduli problem. Since this suffices for the purposes of this paper, only this special case is considered.

Definition 1.2. The moduli functor of stable Gorenstein surfaces is a functor from schemes to sets which assigns to a scheme B the set of isomorphism classes of flat, proper morphisms $X \rightarrow B$ whose fibers are Gorenstein schemes with slc singularities and whose relative dualizing sheaf $\omega_{X/B}$ is ample.

This article considers a smaller functor. Let M_{g_1, g_2} be the functor which assigns to B the set of isomorphism classes of flat proper morphisms $X \rightarrow B$ whose fibers are products of stable curves of genera g_1 and g_2 . The results of this article will show that this functor is coarsely representable by a connected and projective variety and that it is an open and closed subfunctor of the moduli functor of stable Gorenstein surfaces.

2 Deformations of products

In this section, some general deformation-theoretic results are proved about products of varieties. These results are formal and primarily homological. The goal is to show that under some conditions on singularities, the small deformations of a product of varieties are obtained by deforming the factors.

Let $X = Y_1 \times Y_2$ be a variety which is the product of two local complete intersection varieties Y_1 and Y_2 of general type; let π_i denote the projection map to Y_i . This notation will be fixed throughout this section.

The following “rigidity lemma” will be useful:

Lemma 2.1. *If $h : X_1 \times X_2 \rightarrow B_1 \times B_2$ is a surjective morphism of products of stable varieties, then after possibly renumbering, h can be written as the product of maps $h_i : X_i \rightarrow B_i$, $i = 1, 2$.*

Proof. This follows from the fact that the tangent space to the scheme $\text{Hom}(X_i, B_j)$ at the equivalence class $[f]$ of a morphism is $H^0(X_i, f^* \mathcal{T}_{B_j})$ which vanishes due to the stability assumption. A morphism h as in the hypothesis which is not a product would be a non-trivial deformation of some morphism $f : X_i \rightarrow B_j$. \square

In particular, it follows that, up to renumbering, a product of curves of general type can be written as a product of curves in a unique way. This depends on the general type assumption, as there exist abelian surfaces which can be written in distinct ways as the product of elliptic curves.

The assumption that the varieties in this section are local complete intersections implies that the space $T^1(X)$ of first-order infinitesimal deformations of such a variety X is given by $\text{Ext}_X^1(\Omega_X, \mathcal{O}_X)$. See [Vis] for details. Without the local complete intersection hypothesis, the sheaf Ω_X is replaced with the cotangent complex and Ext is taken to be the hyperext in the derived category: $T^1(X) = \mathbf{Ext}^1(\mathbf{L}_X, \mathcal{O}_X)$.

Theorem 2.2. *Every first-order deformation of X is the product of a first order deformation of Y_1 with a first order deformation of Y_2 if Y_1 and Y_2 are of general type.*

Proof. Let \mathbf{L}_X , \mathbf{L}_{Y_1} , and \mathbf{L}_{Y_2} denote the cotangent complexes of X , Y_1 , and Y_2 , respectively. Denote by \mathbf{Ext} the hyperext groups. We need to show:

$$\mathbf{Ext}_X^1(\mathbf{L}_X, \mathcal{O}_X) \cong \mathbf{Ext}_{Y_1}^1(\mathbf{L}_{Y_1}, \mathcal{O}_{Y_1}) \oplus \mathbf{Ext}_{Y_2}^1(\mathbf{L}_{Y_2}, \mathcal{O}_{Y_2})$$

by [Ill71], III.1.2.0.

By [Ill71], II.2.2.3,

$$\mathbf{Ext}^1(\mathbf{L}_X, \mathcal{O}_X) \cong \mathbf{Ext}^1(\pi_1^* \mathbf{L}_{Y_1} \oplus \pi_2^* \mathbf{L}_{Y_2}, \mathcal{O}_X) \tag{1}$$

$$\cong \mathbf{Ext}^1(\pi_1^* \mathbf{L}_{Y_1}, \pi_1^* \mathcal{O}_{Y_1}) \oplus \mathbf{Ext}^1(\pi_2^* \mathbf{L}_{Y_2}, \pi_2^* \mathcal{O}_{Y_2}), \tag{2}$$

The following computation finishes the proof:

$$\mathbf{Ext}^1(\pi_1^* \mathcal{L}_{Y_1}, \pi_1^* \mathcal{O}_{Y_1}) \cong H^1[\mathrm{RHom}(\pi_1^* \mathcal{L}_{Y_1}, \pi_1^* \mathcal{O}_{Y_1})] \quad (3)$$

$$\cong H^1[R\Gamma(X, R\mathcal{H}om(\pi_1^* \mathcal{L}_{Y_1}, \pi_1^* \mathcal{O}_{Y_1}))] \quad (4)$$

$$\cong H^1[R\Gamma(X, \pi_1^* R\mathcal{H}om(\mathcal{L}_{Y_1}, \mathcal{O}_{Y_1}))] \quad (5)$$

$$\cong H^1[R\Gamma(Y_2, R\pi_{2*} \pi_1^* R\mathcal{H}om(\mathcal{L}_{Y_1}, \mathcal{O}_{Y_1}))] \quad (6)$$

$$\cong H^1[R\Gamma(Y_2, \mathcal{O}_{Y_2} \otimes R\Gamma(Y_1, R\mathcal{H}om(\mathcal{L}_{Y_1}, \mathcal{O}_{Y_1})))] \quad (7)$$

$$\cong H^1[R\Gamma(Y_2, \mathcal{O}_{Y_2}) \otimes \mathrm{RHom}(\mathcal{L}_{Y_1}, \mathcal{O}_{Y_1})] \quad (8)$$

$$\cong [H^0(R\Gamma(Y_2, \mathcal{O}_{Y_2})) \otimes H^1(\mathrm{RHom}(\mathcal{L}_{Y_1}, \mathcal{O}_{Y_1}))] \quad (9)$$

$$\oplus [H^1(R\Gamma(Y_2, \mathcal{O}_{Y_2})) \otimes H^0(\mathrm{RHom}(\mathcal{L}_{Y_1}, \mathcal{O}_{Y_1}))] \quad (10)$$

$$\cong \mathbf{Ext}^1(\mathcal{L}_{Y_1}, \mathcal{O}_{Y_1}) \oplus [H^1(Y_2, \mathcal{O}_{Y_2}) \otimes \mathrm{Der}(\mathcal{O}_{Y_1}, \mathcal{O}_{Y_1})] \quad (11)$$

The steps are justified as follows: the composition of derived functors rule ([Har66], II.5.3) justifies steps (4), (6), and part of (8). Step (5) follows from the flatness of π_1 using [Har66], II.5.8. Step (7) is [Har66], II.5.12. Step (8) follows from [Har66], II.5.16. Step (9) is the Künneth formula. Step (10) follows from properness of Y_2 and [Ill71], II.1.2.4.3. Step (11) follows from the fact that varieties of general type have no infinitesimal automorphisms, so $\mathrm{Der}(\mathcal{O}_{Y_1}, \mathcal{O}_{Y_1})$ vanishes. \square

Note that the general type hypothesis is essential: suppose the Y_i were both smooth elliptic curves. Then one may replace all of the $\mathrm{Ext}^1(\Omega, \mathcal{O})$ on X and the Y_i with $H^1(\mathcal{T})$. The tangent sheaf of an abelian variety is trivial, so $h^1(Y_1, \mathcal{T}_{Y_1}) + h^1(Y_2, \mathcal{T}_{Y_2}) = h^1(Y_1, \mathcal{O}_{Y_1}) + h^1(Y_2, \mathcal{O}_{Y_2}) = 2$, but by Hodge theory, $h^1(X, \mathcal{T}_X) = 2h^1(X, \mathcal{O}_X) = \dim_{\mathbf{C}} H^1(X, \mathbf{C}) = 4$.

Corollary 2.3. *The Kuranishi space of a product of finitely many stable curves is smooth.*

Proof. This follows from the fact that the deformations of stable curves are unobstructed, and from the above result shows that the only infinitesimal deformations of the product come from the factors, and are consequently unobstructed. \square

3 Products of stable varieties

An essential advantage in using the compactified moduli space becomes apparent when one can determine all of the stable degenerations of a class of varieties. This section is dedicated to proving that products of smooth curves degenerate to products of stable curves, although the result is slightly more general. Higher-dimensional versions of such a result would depend on a deeper study of obstructions which appear. The proof given here also uses the normality of the moduli spaces of stable curves, which follows from unobstructedness.

Proposition 3.1. *The product of stable curves is a stable surface. More specifically, the product of stable curves has only normal crossings and degenerate cusps as singular points.*

Proof. The question is analytically local. The singularities of stable curves are nodes. Since the product of a smooth point on a curve with a node is a normal crossing singularity, it suffices to check that the product of a node with itself is slc.

One must compute a semiresolution of the scheme $\mathrm{Spec} \mathbf{C}[x, y, w, z]/(xy, wz)$. This scheme is the affine cone over a cycle of rational curves, so blowing up the cone point is a semiresolution with exceptional locus begin a cycle of rational curves. Therefore the singularity at the cone point is a degenerate cusp. All of the other singular points are plainly normal crossings.

Having checked that the singularities are slc, the stability assertion is simply the ampleness of the canonical bundle, which follows from the ampleness of the canonical bundles of the factors. \square

This is a special case of the following more general result, but the proof with coordinate rings is retained to see exactly what singularities occur in the case of products of stable curves.

Theorem 3.2. *Let Y_1 and Y_2 be smoothable stable varieties. Then $Y_1 \times Y_2$ is a smoothable stable variety.*

Proof. The ampleness of the canonical class of $Y_1 \times Y_2$ is immediate. The smoothability is clear, since the fibered product of two smoothings will be a smoothing, since products of rational Gorenstein singularities are rational Gorenstein. It remains to verify that products of slc singularities are slc. First, the conditions of \mathbf{Q} -Gorenstein, S_2 and normal crossings in codimension 1 are clearly preserved under taking products. Let $f : X \rightarrow Y_1$ be a desingularization. Then write

$$K_X = f^* K_{Y_1} + \sum a_i E_i$$

where the E_i are exceptional. The a_i are all greater than or equal to -1 since Y_1 is slc. Therefore the exceptional divisors of the product morphism $X \times Y_2 \rightarrow Y_1 \times Y_2$ occur with coefficient greater than or equal to -1. Since X is smooth and Y_2 is slc, $X \times Y_2$ is slc. Therefore the discrepancies of a resolution of $X \times Y_2$ are all greater than or equal to -1, so $Y_1 \times Y_2$ is slc, since a resolution of $X \times Y_2$ is also a resolution of $Y_1 \times Y_2$. \square

A stronger version of this theorem which depends on minimal model hypotheses, and which we will not use here is in [vO03]. Precisely, the total space of a flat family over a base with only slc singularities whose special fiber has only slc singularities has only slc singularities.

4 Main results

The main theorems below are stated and proved in the case of the product of two surfaces for ease of notation. However, the proofs generalize to the product of finitely many curves. Denote by M_g the moduli functor of stable curves of genus g . This functor is known to be coarsely representable by a projective variety.

Theorem 4.1. *Let $g_1, g_2 \geq 2$. If $g_1 \neq g_2$, then M_{g_1, g_2} is isomorphic to $M_{g_1} \times M_{g_2}$.*

Proof. Taking fibered products gives a natural transformation $M_{g_1} \times M_{g_2} \rightarrow M_{g_1, g_2}$. This natural transformation is relatively representable. By 2.2, it is étale. By 2.1 it is injective on geometric points, that is $M_{g_1}(k) \times M_{g_2}(k) = M_{g_1, g_2}(k)$ when k is an algebraically closed field. The natural transformation is proper since $M_{g_1} \times M_{g_2}$ is proper. It follows that the functors are isomorphic and that M_{g_1, g_2} is coarsely representable. \square

A similar argument proves:

Theorem 4.2. *$M_{g, g}$ is isomorphic to the symmetric square of the functor M_g if $g \geq 2$.*

Corollary 4.3. *Assume the minimal model program. Let $n > 1$. Given $m > 0$, there exists a Hilbert polynomial such that the moduli space of stable Gorenstein varieties of dimension n with this Hilbert polynomial has at least m components.*

Proof. Given $m > 0$, there exists a positive integer N which factors in at least m distinct ways as a product of two distinct factors. Choose m pairs (a_i, b_i) such that $(a_i - 1)(b_i - 1) = N$. Let C_{a_i} and C_{b_i} be smooth curves of genus a_i and b_i , respectively for each i .

Let g_1, \dots, g_{n-2} be distinct integers greater than 1 which are also distinct from all of the a_i and b_i and for each $j = 1, \dots, n-2$, let C_{g_j} be a smooth curve of genus g_j . Then the products

$$\begin{array}{c} C_{g_1} \times \cdots \times C_{g_{n-2}} \times C_{a_1} \times C_{b_1} \\ \vdots \\ C_{g_1} \times \cdots \times C_{g_{n-2}} \times C_{a_m} \times C_{b_m} \end{array}$$

have the same numerical invariants, since these can be computed from the invariants of the curves, and for a product of two curves, χ and K^2 are both multiples of $(a_i - 1)(b_i - 1)$. However, these curves belong to different components of the moduli space since the genera chosen are distinct. \square

One could also draw several easy corollaries of the theorem from the deep results in [HM98] concerning the moduli spaces of stable curves; in particular:

Corollary 4.4. *M_{g_1, g_2} is of general type if g_1 and g_2 are distinct and both greater than 23.*

Also, the rational Picard group is not as simple as in the case of curves.

Corollary 4.5. *Let g_1 and g_2 be distinct integers greater than 2. Then $\text{Pic } M_{g_1, g_2} \otimes \mathbf{Q} \cong (\text{Pic } M_{g_1} \otimes \mathbf{Q}) \times (\text{Pic } M_{g_2} \otimes \mathbf{Q})$.*

Proof. The moduli spaces of curves are integral schemes of finite type. Furthermore, $H^1(M_g, \mathbf{C}) = 0$ (see, e.g. [AC98]). Their singularities are at worst finite quotient singularities, since the Kuranishi spaces for curves are smooth and stable curves have a finite automorphism group. Since finite quotient singularities are DuBois, the results of [DB81] imply that $H^1(M_g, \mathbf{C}) \rightarrow H^1(M_g, \mathcal{O}_{M_g})$ is surjective, so the latter group is zero. The result that the Picard group of the product decomposes as the product of Picard groups under these hypotheses is [Har77] ex. III.12.6. \square

Specifically, for the moduli spaces of curves, the rational Picard group is freely generated by the Hodge class and the classes of the components of the boundary divisor [AC87]. The rational Picard group of the product has too high a rank for the same to be true. This is not surprising, since the cycle structure of the moduli spaces of surfaces is not as simple as that for curves. In general, the boundary is not likely a divisor, and there will be other “geometric” classes which occur, for example, the closure of the locus of surfaces whose canonical model has rational double points.

References

- [AC87] E. Arbarello and M. Cornalba. The Picard groups of the moduli spaces of curves. *Topology*, 26(2):153–171, 1987.
- [AC98] E. Arbarello and M. Cornalba. Calculating cohomology groups of moduli spaces of curves via algebraic geometry. *Inst. Hautes Études Sci. Publ. Math.*, (88):97–127 (1999), 1998.
- [Cat00] F. Catanese. Fibred surfaces, varieties isogenous to a product and related moduli spaces. *Amer. J. Math.*, 122(1):1–44, 2000.
- [DB81] Ph. Du Bois. Complexe de de Rham filtré d’une variété singulière. *Bull. Soc. Math. France*, 109(1):41–81, 1981.
- [HM98] J. Harris and I. Morrison. *Moduli of curves*, volume 187 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.
- [Har77] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.
- [Har66] R. Hartshorne. *Residues and duality*. Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20. Springer-Verlag, Berlin, 1966.
- [Ill71] L. Illusie. *Complexe cotangent et déformations. I*. Springer-Verlag, Berlin, 1971. Lecture Notes in Mathematics, Vol. 239.
- [Kol90] J. Kollár. Projectivity of complete moduli. *J. Differential Geom.*, 32(1):235–268, 1990.
- [KSB88] J. Kollár and N. I. Shepherd-Barron. Threefolds and deformations of surface singularities. *Invent. Math.*, 91(2):299–338, 1988.
- [Vis] A. Vistoli. The deformation theory of local complete intersections. arXiv:alg-geom/9703008.
- [vO03] M. van Opstall. Ph.D. thesis. In preparation.